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Bohman–Korovkin–Wulbert Operators on Normed Spaces

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We introduce a class of bounded linear operators on normed spaces satisfying a Bohman–Korovkin–Wulbert type approximation theorem and investigate a class of such operators on special function spaces. © 1993 Academic Press, Inc.

1. INTRODUCTION

Recently a class of operators on $C[0, 1]$ satisfying a Bohman–Korovkin–Wulbert type theorem was introduced and investigated by the author (cf. [8, 9]). The purpose of this paper is to introduce and to investigate a class of such operators on normed spaces.

In 1952, H. Bohman [2] showed that a sequence of special interpolation operators $\{B_n : n = 1, 2, ...\}$ on $C[0, 1]$ converges strongly to the identity operator on $C[0, 1]$ if $\{B_n(x^m) : n = 1, 2, ...\}$ converges uniformly to the function x^m for each $m = 0, 1, 2$. Such functions $x^m : t \to t^m$ ($t \in [0, 1]$) are called test functions. In 1959, P. P. Korovkin [4] showed that Bohman's theorem is true even if the interpolation operators B_n are replaced with positive linear operators on $C[0, 1]$. In 1968, D. E. Wulbert [10] showed that Korovkin's theorem is true even if the positivity condition on the operators is replaced with the operator norm condition on which the sequence of operator norms converges to one.

Let X and Y be normed spaces and $B(X, Y)$ the Banach space of all bounded linear operators of X into Y. For $S \subset X$, $\mathfrak{B} \subset B(X, Y)$, and $x \in X$, let $\mathrm{BKW}(X, Y; S, \mathfrak{B}, x)$ be the set of all $T \in B(X, Y)$ such that if $\{T_\lambda : \lambda \in \Lambda\}$ is a net in $\mathfrak{B}$ such that $\lim \|T_\lambda\| = \|T\|$ and $\lim \|T_\lambda s - Ts\| = 0$ ($\forall s \in S$), then $\lim \|T_\lambda x - Tx\| = 0$. We further set

$$\mathrm{BKW}(X, Y; S, \mathfrak{B}) = \bigcap_{x \in X} \mathrm{BKW}(X, Y; S, \mathfrak{B}, x).$$

We call an element of $\mathrm{BKW}(X, Y; S, \mathfrak{B})$ a Bohman–Korovkin–Wulbert operator (shortly BKW-operator) of X into Y for the test set S

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and $\mathfrak{B}$. Also set $\mathrm{BKW}(X; S, \mathfrak{B}) = \mathrm{BKW}(X, X; S, \mathfrak{B})$, $\mathrm{BKW}(X, Y; S) = \mathrm{BKW}(X, Y; S, B(X, Y))$, and $\mathrm{BKW}(X, S) = \mathrm{BKW}(X, X, S, B(X, X))$. In this setting, Bohman's theorem (Korovkin's theorem and Wulbert's theorem) asserts that the identity operator on $C[0, 1]$ is a BKW-operator of $C[0, 1]$ into itself for the test functions $\{1, x, x^2\}$ and the family of special interpolation operators on $C[0, 1]$ (the family of positive linear operators on $C[0, 1]$ and the family of bounded linear operators on $C[0, 1]$, respectively).

Let us pick up the known results concerning BKW-operators.

(1) Let $C(\Omega)$ be the Banach space of continuous complex-valued functions on a compact Hausdorff space Ω, X a linear subspace of $C(\Omega)$, and $S \subset X$ a function space on Ω whose Choquet boundary equals Ω. Then Wulbert showed that the inclusion map of X into $C(\Omega)$ belongs to $\mathrm{BKW}(X, C(\Omega); S)$ (cf. [10, Corollary 2]). In particular if Γ is the unit circle and X is a linear subspace of $C(\Gamma)$ which contains $\{1, z, z^-\}$, then the inclusion map of X into $C(\Gamma)$ belongs to $\mathrm{BKW}(X, C(\Gamma); \{1, z, z^-\})$ (cf. [10, Corollary 4]).

(2) Let A be a C^*-algebra and $P^-(A)$ the pure state space of A. Let S be a subset of A such that every $f \in P^-(A)$ is the only positive linear functional on A which extends $f \mid S$. Then the author proved that if A is unital and $1 \in S$, then the identity operator on A is BKW for S and the family of positive linear operators on A (cf. [7, Theorem 3.4]). Furthermore, F. Altomare showed that if B is another C^*-algebra, then every $(P^-(A), P^-(B))$-admissible operator of A into B is BKW for S and the family of positive linear operators of A into B (cf. [1, Theorem 5.1]).

(3) If T is a homomorphism of $C[0, 1]$ into itself and S is an isometric multiplication operator on $C[0, 1]$, then TS and ST belong to $\mathrm{BKW}(C[0, 1]; \{1, x, x^2\})$ (cf. [8, Theorem]).

(4) The sum of two homomorphisms of $C[0, 1]$ into itself belongs to $\mathrm{BKW}(C[0, 1]; \{1, x, x^2, x^3, x^4\})$ (cf. [9, Theorem]).

But, very recently, S. Romanelli [6] characterized the T-universal Korovkin spaces in the context of commutative Banach algebras and observed that (4) is contained in her approximation theory.

In this paper we will give some new BKW-operators on function spaces.

2. Theorems

Let D^- be the closed unit disk. $A(D^-)$ the disk algebra, and $C(\Gamma)$ the Banach space of continuous complex-valued functions on Γ, the unit circle.

THEOREM 1. *Let* $B_1, ..., B_n$ *be finite Blaschke products and* $T_{B_1}, ..., T_{B_n}$ *the homomorphisms of* $A(D^-)$ *into itself defined by* $B_1, ..., B_n$, *respectively. Let* $a_1, ..., a_n$ *be positive real numbers. Then* $a_1 T_{B_1} + \cdots + a_n T_{B_n} \in \mathrm{BKW}(A(D^-); \{1, z, ..., z^n\})$.

Remark. Let T be a nonzero homomorphism of $A(D^-)$ into itself. Set $\phi = T(z)$. Then $\phi \in A(D^-)$ and $(Tf)(z) = f(\phi(z))$ $(\forall z \in D^-, \forall f \in A(D^-))$. If $\phi(\Gamma) \subset \Gamma$, then ϕ must be a finite Blaschke product.

THEOREM 2. *Let* $T_1, ..., T_n$ *be nonzero homomorphisms of* $C(\Gamma)$ *into itself and* $a_1, ..., a_n$ *positive real numbers. Then* $a_1 T_1 + \cdots + a_n T_n \in \mathrm{BKW}(C(\Gamma); \{1, z, ..., z^n\})$.

Given a topological space Ω, we denote by $C^b(\Omega)$ the Banach space of all bounded continuous complex-valued functions on Ω with supremum norm and usual operations.

THEOREM 3. *Let* Ω, Φ *be two topological spaces,* X *a linear subspace of* $C^b(\Omega)$ *such that* $\sup\{|x(\omega)| : x \in X, \|x\| = 1\} = 1$ *for any* $\omega \in \Omega$, Y *a linear subspace of* $C^b(\Phi)$. *Let* S *be a subset of* X *such that*

$$\Omega = \{\omega \in \Omega : f \in X^* \ (\text{the dual space of } X),\ \|f\| \leqslant 1,$$
$$s(\omega) = f(s)\ (\forall s \in S) \Rightarrow x(\omega) = f(x)\ (\forall x \in X)\}$$

and Ψ *a compact subset of* Φ *such that* $\|y\| = \sup\{|y(\phi)| : \phi \in \Psi\}$ *for each* $y \in Y$ *(i.e.,* Ψ *is a boundary of* Y*). Then a norm one linear operator* T *of* X *into* Y *which satisfies the condition*

$$\exists \tau: \Psi \to \Omega,\ \exists u: \Psi \to \Gamma: (Tx)(\phi) = u(\phi)\, x(\tau(\phi)) \qquad (\forall \phi \in \Psi,\ \forall x \in X)$$

belongs to $\mathrm{BKW}(X, Y; S)$.

Remark. Both (1) and (3) in Section 1 are special cases of the above theorem. Also the condition in the above theorem concerns the decomposition theorem of W. Holsztynski [3].

THEOREM 4. *Let* T_1 *and* T_2 *be nonzero homomorphisms of* $C[0, 1]$ *into itself and let* $\alpha(t)$ *and* $\beta(t)$ *be representing functions of* T_1 *and* T_2, *respectively, i.e.* $\alpha(t) = (T_1 x)(t)$, $\beta(t) = (T_2 x)(t)$ $(\forall t \in [0, 1])$, *where* $x(t) = t$ $(\forall t \in [0, 1])$. *Let* $G = \{(\alpha(t), \beta(t)) \in [0, 1]^2 : t \in [0, 1]\}$ *and* $D = \{(t, s) \in [0, 1]^2 : s \geqslant 3t,\ 3s \geqslant t + 2\} \cup \{(t, s) \in [0, 1]^2 : t \geqslant 3s,\ 3t - 2 \geqslant s\}$. *If* $G \subset D$ *and if* a *and* b *are positive numbers, then* $aT_1 - bT_2 \in \mathrm{BKW}(C[0, 1]; \{1, x, x^2, x^3\})$.

3. Basic Lemma

Let X be a normed space and X^* its dual space. For $S \subset X$ and $x \in X$, let $U(X, S; x)$ be the set of all $f \in X^*$ such that if g is an element of X^* satisfying $\|g\| \leqslant \|f\|$ and $g \mid S = f \mid S$, then $g(x) = f(x)$. We further set

$$U(X, S) = \bigcap_{x \in X} U(X, S; x).$$

The following lemma is the main tool in our approximation theory.

Lemma. *Let X, Y be normed spaces, $x \in X$, and $S \subset X$. Let E be a weak*-closed subset of Y^* such that $\|g\| = 1$ for all $g \in E$. Let T be an operator in $B(X, Y)$ such that $\|T\| = \|T^*g\|$ and $T^*g \in U(X, S; x)$ for all $g \in E$, where T^* is the dual map of T. If $\{T_\lambda\}$ is a net in $B(X, Y)$ such that $\lim \|T_\lambda\| = \|T\|$ and $\lim_\lambda \sup_{g \in E} |g(T_\lambda s) - g(Ts)| = 0$ for all $s \in S$, then $\lim_\lambda \sup_{g \in E} |g(T_\lambda x) - g(Tx)| = 0$.*

This can be directly proved by a similar method to that used in the proof of [5, 8]. However, we show this as an application of the following result of Altomare which is the main tool of his approximation theory in [1].

Theorem A [1, Corollary 1.2]. *Let X be a topological linear space and X^* its topological dual with weak*-topology. Let A be an equicontinuous subset of X^*, B a weak*-closed subset of X^*, and S a subset of X. Let $W_u(X, S, A, B)$ be the linear subspace of all $x \in X$ which verifies the following property*:

> *if Y is a linear topological space, C an equicontinuous weak*-closed subset of Y^*, D a weak*-closed subset of Y^* with $C \subset D$, $\{T_\lambda\}$ an equicontinuous net of (B, D)-admissible linear maps from X to Y, and $T: X \to Y$ an (A, C)-admissible linear map such that $\lim_\lambda \sup_{g \in C} |g(T_\lambda s) - g(Ts)| = 0$ $(\forall s \in S)$, then $\lim_\lambda \sup_{g \in C} |g(T_\lambda x) - g(Tx)| = 0$.*

Then $W_u(X, S, A, B)$ equals the set of all $x \in X$ such that if $f \in A$ and $g \in B$ and if $f(s) = g(s)$ for all $s \in S$, then $f(x) = g(x)$.

Proof of the Lemma. Let $\{T_\lambda\}$ be a net of $B(X, Y)$ such that $\lim \|T_\lambda\| = \|T\|$ and $\lim_\lambda \sup_{g \in E} |g(T_\lambda s) - g(Ts)| = 0$ for all $s \in S$. In this case we can assume without loss of generality that $\|T_\lambda\| = \|T\| = 1$ for all λ. Now set

$$\begin{aligned} A &= \{f \in U(X, S; x) : \|f\| = 1\}, \\ B &= \{f \in X^* : \|f\| \leqslant 1\}, \\ C &= E, \\ D &= \{g \in Y^* : \|g\| \leqslant 1\}. \end{aligned}$$

Then $\{T_\lambda\}$ is an equicontinuous net of (B, D)-admissible linear maps from X to Y. If also $f \in A$ and $g \in B$ and if $f(s) = g(s)$ for all $s \in S$, then $\|g\| \leqslant 1 = \|f\|$ and hence $f(x) = g(x)$. Therefore by Theorem A, $x \in W_u(X, S, A, B)$ and hence if $T \in B(X, Y)$ is such that $\|T\| = \|T^*g\|$ and $T^*g \in U(X, S; x)$ for all $g \in E$, then T is (A, C)-admissible and so $\lim_\lambda \sup_{g \in E} |g(T_\lambda x) - g(Tx)| = 0$. ∎

Remark. $U(X, S; x) \subset \mathrm{BKW}(X, \mathbb{C}; S, x) \subset U^1(X, S; x) := \{f \in X^* : g \in X^*, \|g\| = \|f\|, g \mid S = f \mid S \Rightarrow g(x) = f(x)\}$. Here $\mathbb{C}$ is the complex numbers.

4. Proofs of Theorems

(1) *Proof of Theorem* 1. We will identify D^- and Γ with the carrier space of $A(D^-)$ and the Silov boundary of D^- with respect to $A(D^-)$, respectively. Now let $B_1, ..., B_n$ be finite Blaschke products and $T_{B_1}, ..., T_{B_n}$ the homomorphisms of $A(D^-)$ into itself defined by $B_1, ..., B_n$, respectively. Also let $a_1, ..., a_n$ be positive real numbers and set $T = a_1 T_{B_1} + \cdots + a_n T_{B_n}$. We further set $S = \{1, z, ..., z^n\}$ and $E = \Gamma$. Hence, if we can show that

(i) $\|T\| = \|T^* z_0\|$ and

(ii) $T^* z_0 \in U(A(D^-), S)$

for all $z_0 \in E$, then we obtain that $T \in \mathrm{BKW}(A(D^-); S)$ by the basic lemma. Thus we only need to show (i) and (ii). Let $z_0 \in E$ be fixed. Note that $T^* z_0 = a_1 B_1(z_0) + \cdots + a_n B_n(z_0)$ and hence $\|T^* z_0\| = a_1 + \cdots + a_n$. Therefore we have

$$\|T\| = \left\| \sum_{j=1}^{n} a_j T_{B_j} \right\| \leqslant \sum_{j=1}^{n} a_j = \|T^* z_0\| \leqslant \|T^*\| = \|T\|,$$

so that (i) was shown. Next to show (ii), let $\nu \in (A(D^-))^*$ be such that $T^* z_0 \mid S = \nu \mid S$ and $\|\nu\| \leqslant \|T^* z_0\|$. So we want to show that $\nu = T^* z_0$. Since we can regard $A(D^-)$ as a closed subspace of $C(\Gamma)$, we can take a linear functional μ on $C(\Gamma)$ such that $\|\mu\| = \|\nu\|$ and $\mu \mid A(D^-) = \nu$ by the Hahn–Banach extension theorem. Set

$$z_1 = B_1(z_0), ..., z_n = B_n(z_0).$$

Then $z_1, ..., z_n \in \Gamma$ and $a_1 \delta_{z_1} + \cdots + a_n \delta_{z_n}$ is a norm preserving linear extension of $T^* z_0$ to $C(\Gamma)$. Here δ_z denotes the evaluation at $z \in \Gamma$. Hence we have that $\mu \mid S = a_1 \delta_{z_1} + \cdots + a_n \delta_{z_n} \mid S$. Put

$$z_1 = \exp(i\theta_1), ..., z_n = \exp(i\theta_n),$$

where $i = \sqrt{-1}$. Then we obtain that

$$\mu(z^k) = \sum_{j=1}^{n} a_j \exp(ik\theta_j) \qquad (k = 0, 1, ..., n).$$

In particular,

$$\|\mu\| = \|\nu\| \leqslant \|T^* z_0\| = a_1 + \cdots + a_n = \mu(1) \leqslant \|\mu\|,$$

and hence μ can be regarded as a positive measure on Γ. Now let τ be the isomorphism of $C(\Gamma)$ onto $C_{2\pi}(\mathbb{R})$ defined by

$$\tau : f(z) \to f(\exp(i\theta)) \qquad (z = \exp(i\theta)),$$

where $C_{2\pi}(\mathbb{R})$ is the Banach space of continuous complex-valued functions on the real numbers $\mathbb{R}$ with period 2π. Set $\mu^{\sim} = (\tau^*)^{-1}(\mu)$, where τ^* is the dual map of τ. Then $\mu^{\sim}$ is a positive linear functional on $C_{2\pi}(\mathbb{R})$. Also since $\mu^{\sim}(\exp(ik\theta)) = \mu(z^k)$ $(k = 0, 1, 2, \ldots)$, it follows that

$$\mu^{\sim}(\cos k\theta) = \sum_{j=1}^{n} a_j \cos k\theta_j$$

and

$$\mu^{\sim}(\sin k\theta) = \sum_{j=1}^{n} a_j \sin k\theta_j$$

for all $k = 0, 1, \ldots, n$. Put

$$F(\theta) = \prod_{j=1}^{n} \sin^2 \frac{\theta - \theta_j}{2}.$$

Then F belongs to the linear subspace of $C_{2\pi}(\mathbb{R})$ generated by $\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}$ and so it can be written as

$$F(\theta) = \sum_{k=0}^{n} b_k \cos k\theta + c_k \sin k\theta.$$

Therefore we have

$$\begin{aligned}
\mu^{\sim}(F) &= \sum_{k=0}^{n} b_k \mu^{\sim}(\cos k\theta) + c_k \mu^{\sim}(\sin k\theta) \\
&= \sum_{k=0}^{n} \left\{ b_k \sum_{j=1}^{n} a_j \cos k\theta_j + c_k \sum_{j=1}^{n} a_j \sin k\theta_j \right\} \\
&= \sum_{j=1}^{n} a_j \left\{ \sum_{k=0}^{n} (b_k \cos k\theta_j + c_k \sin k\theta_j) \right\} \\
&= \sum_{j=1}^{n} a_j F(\theta_j) \\
&= 0.
\end{aligned}$$

Therefore the support of $\mu^{\sim}$ is contained in $\{\theta_1, ..., \theta_n\}$ (mod 2π) and so we can write

$$\mu^{\sim} = \sum_{j=1}^{n} \alpha_j \delta_{\theta_j}.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{n} a_j z_j^k = \sum_{j=1}^{n} a_j \exp(ik\theta_j) = \mu(z^k) = \mu^{\sim}(\exp(ik\theta)) \\ = \sum_{j=1}^{n} \alpha_j \exp(ik\theta_j) = \sum_{j=1}^{n} \alpha_j z_j^k\end{aligned}$$

for all $k = 0, 1, ..., n$, so we have

$$\begin{aligned}\mu(f) &= \sum_{j=1}^{n} \alpha_j f(\exp(i\theta_j)) = \sum_{j=1}^{n} \alpha_j f(z_j) = \sum_{j=1}^{n} a_j f(z_j) \\ &= \sum_{j=1}^{n} a_j \delta_{z_j}(f)\end{aligned}$$

for all $f \in C(\Gamma)$. Consequently we have that

$$\nu = \mu \mid A(D^-) = \sum_{j=1}^{n} a_j \delta_{z_j} \mid A(D^-) = T^* z_0$$

and so (ii) is shown. ∎

(2) *Proof of Theorem* 2. This can be easily shown by a similar method to that used in the above proof. ∎

(3) *Proof of Theorem* 3. Set $E = \{\delta_\phi \mid Y : \phi \in \Psi\}$, where δ_ϕ is the evaluation at ϕ. Then by the compactness of Ψ, E is a weak*-compact subset of Y^*. Also since

$$\langle x, T^*(\delta_\phi \mid Y)\rangle = u(\phi)\, x(\tau(\phi)) = \langle x, u(\phi)(\delta_{\tau(\phi)} \mid X)\rangle$$

for all $\phi \in \Psi$ and for all $x \in X$, we have

$$T^*(\delta_\phi \mid Y) = u(\phi)\, \delta_{\tau(\phi)} \mid X$$

for all $\phi \in \Psi$. Therefore $\|T^*(\delta_\phi \mid Y)\| = 1$ for all $\phi \in \Psi$. Since T is norm one, we obtain that $\|T^* g\| = \|T\|$ for all $g \in E$. Suppose next that $\mu \in X^*$,

$\|\mu\| \leqslant 1$, $\phi \in \Psi$, and $\langle s, T^*(\delta_\phi \mid Y)\rangle = \mu(s)$ $(\forall s \in S)$. Set $\omega = \tau(\phi)$ and $f = u(\phi)^{-1}\mu$. Then $\|f\| \leqslant 1$. Also since

$$\begin{aligned} s(\omega) &= u(\phi)^{-1}(Ts)(\phi) = u(\phi)^{-1}\langle s, T^*(\delta_\phi \mid Y)\rangle \\ &= u(\phi)^{-1}\mu(s) = f(s) \end{aligned}$$

for all $s \in S$, we obtain by the assumption on S that $x(\omega) = f(x)$ for all $x \in X$, namely, $\mu = T^*(\delta_\phi \mid Y)$. In other words, $T^*(g) \in U(X, S)$ for all $g \in E$. Therefore since Ψ is a boundary of Y, $T \in \mathrm{BKW}(X, Y; S)$ follows from the basic lemma. ∎

Proof of Theorem 4. Suppose that $G \subset D$ and let a and b be positive numbers. Set $T = aT_1 - bT_2$. We further set $S = \{1, x, x^2, x^3\}$ and $E = \{\delta_t : t \in [0, 1]\}$. Hence, if we can show that

(i) $\|T\| = \|T^*g\|$ and

(ii) $T^*g \in U(C[0, 1], S)$

for all $g \in E$, then we obtain that $T \in \mathrm{BKW}(C[0, 1]; S)$ by the basic lemma. Thus we only need to show (i) and (ii). Let $t \in [0, 1]$ be fixed and set $\alpha = \alpha(t)$, $\beta = \beta(t)$. Then $\alpha, \beta \in [0, 1]$ and $T^*\delta_t = a\delta_\alpha - b\delta_\beta$, so that $\|T^*\delta_t\| = a + b$. Therefore

$$\|T\| \leqslant a\,\|T_1\| + b\,\|T_2\| = a + b = \|T^*\delta_t\| \leqslant \|T\|,$$

so (i) is shown. To show (ii) let $\mu \in C[0, 1]^*$ be such that $\|\mu\| \leqslant a + b$ and $\mu \mid S = T^*\delta_t \mid S$ (and hence $\mu(x^m) = a\alpha^m - b\beta^m$ $(m = 0, 1, 2, 3)$). Then we must show that $\mu = T^*\delta_t$. We may identify $C[0, 1]^*$ with the measure space $M[0, 1]$ of bounded regular Borel measures on $[0, 1]$.

Suppose first that μ is a real measure and let $\mu = \mu^+ - \mu^-$ be the Hahn decomposition of μ. Then $\|\mu^+\| + \|\mu^-\| \leqslant a + b$ and $\|\mu^+\| - \|\mu^-\| = \mu(1) = a - b$, so that $\|\mu^+\| \leqslant a$ and $\|\mu^-\| \leqslant b$. We can assume without loss of generality that $\alpha < \beta$. Then $(\alpha, \beta) \in \{(t, s) \in [0, 1]^2 : s \geqslant 3t,\ 3s \geqslant t + 2\}$ by the assumption $G \subset D$. Set

$$f(u) = \frac{1}{3}(u^3 - \beta^3) - \frac{\alpha + \beta}{2}(u^2 - \beta^2) + \alpha\beta(u - \beta) \qquad (\forall u \in [0, 1]).$$

Then $f \in \langle S\rangle$, the linear hull of S, $f(\beta) = 0$, and $0 \leqslant f(u) \leqslant f(\alpha)$ $(\forall u \in [0, 1])$. Therefore

$$\begin{aligned} 0 \leqslant \mu^-(f) &= \mu^+(f) - \mu(f) = \mu^+(f) - af(\alpha) \\ &\leqslant \|\mu^+\| f(\alpha) - af(\alpha) \leqslant 0, \end{aligned}$$

so $\mu^-(f) = 0$ and hence the support of μ^- is contained in $\{0, \beta\}$. Then we can write $\mu^- = b_1\delta_\beta + b_2\delta_0$ for some $b_1, b_2 \geqslant 0$, so that

$$\|\mu^-\| = b_1 + b_2. \tag{1}$$

Assume that $\alpha \neq 0$. Set $g(u) = f(\alpha) - f(u)$ ($\forall u \in [0, 1]$). Then $g \in \langle S \rangle$, $g(\alpha) = 0$, and $0 \leqslant g(u) \leqslant g(\beta)$ ($\forall u \in [0, 1]$). Note also that

$$\mu(g) = (a\delta_\alpha - b\delta_\beta)(g) = -bg(\beta)$$

and

$$\mu(g) = \mu^+(g) - \mu^-(g) = \mu^+(g) - b_1 g(\beta) - b_2 g(0).$$

Therefore

$$\begin{aligned} 0 \leqslant \mu^+(g) &= b_1 g(\beta) + b_2 g(0) - bg(\beta) \\ &\leqslant (b_1 - b)\, g(\beta) + b_2 g(\beta) \\ &\leqslant (\|\mu^-\| - b)\, g(\beta) \\ &\leqslant 0. \end{aligned}$$

so $\mu^+(g) = 0$ and hence the support of μ^+ is contained in $\{\alpha, 1\}$. Then we can write $\mu^+ = a_1 \delta_\alpha + a_2 \delta_1$ for some $a_1, a_2 \geqslant 0$, so that

$$\|\mu^+\| = a_1 + a_2. \tag{2}$$

Also since $\mu = a_1 \delta_\alpha + a_2 \delta_1 - b_1 \delta_\beta - b_2 \delta_0$, we have

$$\alpha a_1 + a_2 - \beta b_1 = a\alpha - b\beta \qquad (= \mu(x)) \tag{3}$$

and

$$\alpha^2 a_1 + a_2 - \beta^2 b_1 = a\alpha^2 - b\beta^2 \qquad (= \mu(x^2)). \tag{4}$$

Then by (2), (3), and (4), we have

$$a_2 = \frac{\alpha\beta(\beta - \alpha)(\|\mu^+\| - a)}{\beta - \beta^2 + \alpha\beta(\beta - \alpha)} \leqslant 0.$$

so $a_2 = 0$ and hence $a_1 = \|\mu^+\| = a$. By (3), $b_1 = b$ and so $b_2 = \|\mu^-\| - b \leqslant 0$ by (1), hence $b_2 = 0$. We thus obtain that $\mu = a\delta_\alpha - b\delta_\beta = T^*\delta_t$. Assume that $\alpha = 0$. Then β is the only zero point of f, so by the above argument, $\mu^- = \|\mu^-\|\, \delta_\beta$. Also by the above argument, we can write $\mu^+ = c_1 \delta_0 + c_2 \delta_1$ for some $c_1, c_2 \geqslant 0$. Then $\mu = c_1 \delta_0 + c_2 \delta_1 - \|\mu^-\|\, \delta_\beta$, so that $-b\beta = c_2 - \|\mu^-\| \beta$ $(= \mu(x))$. Then $c_2 = (\|\mu^-\| - b)\beta \leqslant 0$, so $c_2 = 0$ and $b = \|\mu^-\|$. Then $\mu = c_1 \delta_0 - b\delta_\beta$, so $a - b = \mu(1) = c_1 - b$, hence $a = c_1$. We thus obtain that $\mu = a\delta_0 - b\delta_\beta = T^*\delta_t$.

Suppose next that μ is a complex measure. Let μ_1 and μ_2 be the real part and the imaginary part of μ, respectively. Then $\|\mu_1\| \leqslant \|\mu\| \leqslant a + b$,

$\mu_1(x^m) = a\alpha^m - b\beta^m$, and $\mu_2(x^m) = 0$ $(m = 0, 1, 2, 3)$, so that $\mu_1 = a\delta_\alpha - b\delta_\beta$ follows from the real case. Now let H be the set of all functions $h \in C[0, 1]$ such that $h(\alpha) = 1$, $h(\beta) = -1$, and $-1 \leqslant h(u) \leqslant 1$ $(\forall u \in [0, 1])$. Then for any $h \in H$,

$$\begin{aligned} a + b \geqslant |\mu(h)| &= \sqrt{|ah(\alpha) - bh(\beta)|^2 + |\mu_2(h)|^2} \\ &= \sqrt{|a + b|^2 + |\mu_2(h)|^2} \\ &\geqslant a + b, \end{aligned}$$

so $\mu_2(h) = 0$. Let p be any polynomial with real coefficients on $[0, 1]$ and set

$$q(u) = p(u) - \frac{p(\alpha) - p(\beta)}{\alpha - \beta} u + \frac{\beta p(\alpha) - \alpha p(\beta)}{\alpha - \beta}$$

for each $u \in [0, 1]$ and set $r = q/\|q\|_\infty$. Then $r(\alpha) = r(\beta) = 0$ and $-1 \leqslant r(u) \leqslant 1$ $(\forall u \in [0, 1])$, so we can choose a function $h \in H$ such that $r + h \in H$. Then $\mu_2(r) = \mu_2(r + h) = 0$ and hence $\mu_2(p) = 0$. In other words, $\mu_2 = 0$. Consequently, $\mu = \mu_1 = a\delta_\alpha - b\delta_\beta = T^*\delta_t$ and (ii) is shown. ∎

Remark. Let $\{z_1, ..., z_n\}$ be a finite subset of $\mathbb{C}$ such that $z_i \neq z_j$ $(i \neq j)$, let X be the Banach space of complex-valued functions on $\{z_1, ..., z_n\}$, and set $S = \{1, z, z^2\} \subset X$. If $n \leqslant 3$, then BKW(X, S) equals $B(X)$, all the linear operators on X. But the class of operators in BKW(X, S) which can be obtained by means of the basic lemma is the only class of operators $T \in B(X)$ such that if (t_{ij}) is the representing matrix of T, then $(t_{i1}, ..., t_{in}) \in U(X, S)$ $(i = 1, ..., n)$. Here we regard $(t_{i1}, ..., t_{in})$ as an element of X^*.

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